Akademie věd České republiky
Ústav teorie informace a automatizace

Academy of Sciences of the Czech Republic
Institute of Information Theory and Automation

RESEARCH REPORT

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Stable distributions: On densities

No. 2307 October 2011

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Stable Distributions: On Densities.

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September 26, 2011

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1 Introduction

William Feller stated in [1] that densitites of stable distributions are probably not known. V.M. Zolotarev spent a lot of effort on deriving aproximative formulas via series expansions and integral representations for stable densities. In this report we investigate derivation of stable densities and stable distribution function. We further investigate relation between stable distribution functions and incomplete gamma functions.

We define stable distribution using its characteristic exponent. We use parametrization discussed in Karlová [**?**]. Consider probability distribution F with its characteristic function $\hat{F}$. We will call F stable distribution iff its characteristic function is of the following form:

$$\hat{F}(k) = e^{\psi(k)},$$

where

$$\psi(k) = -c|k|^{\alpha} e^{i\alpha(p_1 - p_2)\frac{\pi}{2} \operatorname{sgn} k}, \tag{1.1}$$

with $c > 0$, $0 \leq p_1, p_2 \leq 1$ and $p_1 + p_2 = 1$.

2 Basic Properties

Let us start with expressing stable density via inverse Fourier transformation:

$$f(x; \alpha, p_1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx - c|k|^{\alpha} e^{-i\alpha(p_1 - p_2)\frac{\pi}{2} \operatorname{sgn} k}} \mathrm{d}k. \tag{2.1}$$

We denote the integrand as $g(k, p_1)$. Next, we show simple, yet very useful, property of stable density:

$$f(x; \alpha, p_1) = f(-x, \alpha, p_2). \tag{2.2}$$

For $x \geq 0$:

$$\begin{aligned} f(x; \alpha, p_1) &= \frac{1}{2\pi} \int_0^{\infty} \left[e^{-ikx - ck^{\alpha} e^{-i\alpha(p_1 - p_2)\frac{\pi}{2}}} + e^{ikx - ck^{\alpha} e^{i\alpha(p_1 - p_2)\frac{\pi}{2}}} \right] \mathrm{d}k = \\ &= \frac{1}{2\pi} \int_0^{\infty} \left[e^{ik(-x) - ck^{\alpha} e^{i\alpha(p_2 - p_1)\frac{\pi}{2}}} + e^{-ik(-x) - ck^{\alpha} e^{-i\alpha(p_2 - p_1)\frac{\pi}{2}}} \right] \mathrm{d}k = \\ &= f(-x; \alpha, p_2). \end{aligned}$$

We conclude that $g(k, x; p_1) + \overline{g(k, x; p_1)} = g(k, -x; p_2) + \overline{g(k, -x; p_2)}$. Therefore without loss of generality we can consider $x \geq 0$ and formulate consequent relation:

$$\begin{aligned} f(x; \alpha, p_1) &= \frac{1}{\pi} \mathrm{Re} \int_0^{\infty} e^{-ikx - ck^{\alpha} e^{-i\alpha(p_1 - p_2)\frac{\pi}{2}}} \mathrm{d}k = \\ &= \frac{1}{\pi} \mathrm{Re} \int_0^{\infty} e^{ikx - ck^{\alpha} e^{i\alpha(p_1 - p_2)\frac{\pi}{2}}} \mathrm{d}k. \end{aligned} \tag{2.3}$$

Let us compute real and imaginary part of the integrand $g(k,x;p_1)$. Assume $k>0$:

$$\begin{aligned} g(k,x;p_1) &= e^{-ikx-ck^\alpha e^{-i\alpha(p_1-p_2)\frac{\pi}{2}}} = \\ &= e^{-ck^\alpha \cos[\alpha(p_1-p_2)\frac{\pi}{2}]}\cos[kx-ck^\alpha\sin[\alpha(p_1-p_2)\pi/2]]- \\ &- ie^{-ck^\alpha \cos[\alpha(p_1-p_2)\frac{\pi}{2}]}\sin[kx-ck^\alpha\sin[\alpha(p_1-p_2)\pi/2]]. \end{aligned}$$

Thus we conclude, that:

$$f(x,\alpha,p_1)=\frac{1}{\pi}\int_0^\infty e^{-ck^\alpha \cos[\alpha(p_1-p_2)\frac{\pi}{2}]}\cos[kx-ck^\alpha\sin[\alpha(p_1-p_2)\pi/2]]\mathrm{d}k. \tag{2.4}$$

From previous, we can compute $f(0,\alpha,1/2)$ by direct integration. We use substitution $t=ck^\alpha$ and obtain:

$$f(0,\alpha,1/2)=\frac{c^{1/\alpha}}{\alpha\pi}\int_0^\infty t^{\frac{1}{\alpha}-1}e^{-t}\mathrm{d}t=\frac{c^{1/\alpha}}{\pi}\Gamma\Big[1+\frac{1}{\alpha}\Big].$$

By assuming $p_1=p_2$, the integrand became substantially simplified. Direct computation of $f(0;\alpha,p_1)$ in case of $p_1\neq p_2$ is, however, considerably more complicated. We make use of following lemma.

Lemma 2.1. *For* $0<\theta<\frac{\pi}{2}$ *and* $1<\alpha<2$*:*

$$\begin{aligned} \int_0^\infty e^{-k^\alpha\cos\theta}\cos[k^\alpha\sin\theta]\mathrm{d}k &= \Gamma\Big[1+\frac{1}{\alpha}\Big]\cos\theta, \\ \int_0^\infty e^{-k^\alpha\cos\theta}\sin[k^\alpha\sin\theta]\mathrm{d}k &= \Gamma\Big[1+\frac{1}{\alpha}\Big]\sin(\theta+\pi). \end{aligned}$$

Preliminarily, we start with notation. Having two contours $L_1:[a_1,b_1]\to\mathbb{C}, L_2:[a_2,b_2]\to\mathbb{C}$ with $L_1(b_1)=L_2(a_2)$, we denote the oriented sum of contours as $L_1\dot{+}L_2$. The orientation of the contour is denoted by $\dot{+}$ and so by $\dot{-}L_2$ we denote contour with oposite orientation to contour L_2. If the countour L is closed, we denote the its interior as $\mathrm{int}L$. Next, we prove the lemma.

Proof. Consider a complex valued function $g(z)=e^{-z^\alpha e^{i\theta}}\mathrm{d}z$, for $z\in\mathbb{C}$. For $0<r_1<r_2<\infty$ we define a contour L as:

$$\begin{aligned} L &= L_1\dot{+}L_2\dot{-}L_3\dot{-}L_4,\ \text{where} \qquad (2.5)\\ L_1(t)&=t \text{ for } t\in[r_1,r_2];\ L_2(t)=r_2e^{it} \text{ for } t\in[0,\theta];\\ L_3(t)&=te^{i\theta} \text{ for } t\in[r_1,r_2];\ L_4(t)=r_1e^{it} \text{ for } t\in[0,\theta]. \end{aligned}$$

Next, we integrate $g(z)$ along contour L:

$$\begin{aligned} \int_L e^{-z^\alpha e^{i\theta}}\mathrm{d}z &= \int_{r_1}^{r_2} e^{-t^\alpha e^{i\theta}}\mathrm{d}t + i\int_0^\theta r_2 e^{it-r_2^\alpha e^{i(\alpha t+\theta)}}\mathrm{d}t- \\ &- \int_{r_1}^{r_2} e^{i\theta}e^{-t^\alpha}\mathrm{d}t - i\int_0^\theta r_1 e^{it-r_1^\alpha e^{i(\alpha t+\theta)}}\mathrm{d}t. \end{aligned}$$

Because contour L is closed and integrand $g(z)$ is regular everywhere in interior of contour L, we can apply Cauchy Theorem and obtain:

$$\int_L e^{-z^\alpha e^{i\theta}} \mathrm{d}z = 0.$$

Next, we estimate values of integrals over arcs L_2 and L_4. Let us start with arc L_2. We make use of relation $|\exp z| = \exp(\mathrm{Re}\ z)$.

$$\Big|\int_0^\theta i r_2 e^{it - r_2^\alpha e^{i(\alpha t+\theta)}} \mathrm{d}t\Big| = r_2 \Big|\int_0^\theta (-\sin t + i\cos t) e^{-r_2^\alpha[\cos(\alpha t+\theta) + i\sin(\alpha t+\theta)]} \mathrm{d}t\Big| \leq$$
$$\leq r_2 \int_0^\theta e^{-r_2^\alpha \cos(\alpha t+\theta)} \mathrm{d}t.$$

Because ...

$$r_2 \int_0^\theta e^{-r_2^\alpha \cos(\alpha t+\theta)} \mathrm{d}t = \frac{r_2}{\alpha} \int_\theta^{(\alpha+1)\theta} e^{-r_2^\alpha \cos(y)} \mathrm{d}y = \frac{r_2}{\alpha} \int_\theta^{(\alpha+1)\theta} e^{-r_2^\alpha \sin(y)} \mathrm{d}y \leq$$
$$\leq \frac{r_2}{\alpha} \int_0^{(\alpha+1)\theta} e^{-r_2^\alpha y} \mathrm{d}y < \frac{1 - e^{-r_2^\alpha(\alpha+1)\theta}}{\alpha r_2^{\alpha-1}} \longrightarrow 0,$$

for $r_2 \to \infty$. We proved that integrals along arcs $L_2,\ L_4$ converges to 0, which implies following equity:

$$\int_0^\infty e^{-t^\alpha e^{i\theta}} \mathrm{d}t = e^{i\theta} \int_0^\infty e^{-t^\alpha} \mathrm{d}t = e^{i\theta} \Gamma\Big[\frac{1}{\alpha} + 1\Big]. \tag{2.6}$$

Finally, we compare real and imaginary part of later equity:

$$\int_0^\infty e^{-t^\alpha \cos\theta} \cos[t^\alpha \sin\theta] \mathrm{d}t = \Gamma\Big[\frac{1}{\alpha} + 1\Big] \cos\theta,$$
$$\int_0^\infty e^{-t^\alpha \cos\theta} \sin[-t^\alpha \sin\theta] \mathrm{d}t = \Gamma\Big[\frac{1}{\alpha} + 1\Big] \sin(\theta + \pi),$$

where we used relation $-\sin\theta = \sin(\theta + \pi)$. Q.E.D

Returning to problem of computation $f(0, \alpha, p_1)$ for any admissible p_1, we use Lemma 1.1 with $\theta = \alpha(p_1 - p_2)\frac{\pi}{2}$ and substitute $t = ck^\alpha$. We obtain:

$$f(0; \alpha, p_1) = \frac{c^{1/\alpha}}{\pi} \Gamma\Big[1 + \frac{1}{\alpha}\Big] \cos[\alpha(p_1 - p_2)\pi/2]. \tag{2.7}$$

The previous discussion served us as a motivation for investigating analytic continuation of integrand $g(k, \alpha, p_1)$ in complex plane. We also realized the importance of choice of suitable contours for evaluation of the integral. Analytic extension and choice of contour are closely contected.

Let us conclude this section with a useful property of stable distribution function. Using (2.2), we obtain:

$$F(x;\alpha,p_1) = \int_{-\infty}^{x} f(y;\alpha,p_1)\mathrm{d}y = \int_{-x}^{\infty} f(y;\alpha,p_2)\mathrm{d}y = \\ = 1 - F(-x;\alpha,p_2),$$

and thus $F(x;\alpha,p_1) + F(-x;\alpha,p_2) = 1$.

3 Integral Representation

A density of stable distribution in the form of integral representation is:
for $x > 0$

$$f(x;\alpha,\beta) = \frac{\alpha x^{1/(\alpha-1)}}{(\alpha-1)\pi} \int\limits_{-\beta/\alpha}^{\pi/2} e^{-x^{\alpha/(\alpha-1)}U(\varphi;\beta)} U(\varphi;\beta)\mathrm{d}\varphi. \tag{3.1}$$

Using the following relation:

$$\frac{\partial}{\partial x}\left[e^{-x^{\alpha/(\alpha-1)}U(\varphi;\beta)}\right] = -\frac{\alpha}{\alpha-1} x^{1/(\alpha-1)} e^{-x^{\alpha/(\alpha-1)}U(\varphi;\beta)} U(\varphi;\beta), \tag{3.2}$$

we have equality:

$$f(x;\alpha,\beta) = \frac{\alpha x^{1/(\alpha-1)}}{(\alpha-1)\pi} \int\limits_{-\beta/\alpha}^{\pi/2} e^{-x^{\alpha/(\alpha-1)}U(\varphi;\beta)} U(\varphi;\beta)\mathrm{d}\varphi = \frac{1}{\pi} \int\limits_{-\beta/\alpha}^{\pi/2} \frac{\partial}{\partial x}\left[-e^{-x^{\alpha/(\alpha-1)}U(\varphi;\beta)}\right]\mathrm{d}\varphi. \tag{3.3}$$

Using the symmetry $f(x;\alpha,\beta) = f(-x;\alpha,-\beta)$, we have for $x < 0$:

$$f(x;\alpha,\beta) = \frac{\alpha |x|^{1/(\alpha-1)}}{(\alpha-1)\pi} \int\limits_{\beta/\alpha}^{\pi/2} e^{-|x|^{\alpha/(\alpha-1)}U(\varphi;-\beta)} U(\varphi;-\beta)\mathrm{d}\varphi. \tag{3.4}$$

For $x = 0$:

$$f(0;\alpha,\beta) = \frac{1}{\pi}\Gamma(1+1/\alpha)\cos(\beta/\alpha). \tag{3.5}$$

Let us verify that $\int_{-\infty}^{\infty} f(x;\alpha,\beta)\mathrm{d}x = 1$:

$$\int_{-\infty}^{\infty} f(x;\alpha,\beta)\mathrm{d}x = \int_{0}^{\infty} f(x;\alpha,\beta)\mathrm{d}x + \int_{-\infty}^{0} f(x;\alpha,\beta)\mathrm{d}x = \int_{0}^{\infty} f(x;\alpha,\beta)\mathrm{d}x + \int_{-\infty}^{0} f(-x;\alpha,-\beta)\mathrm{d}x = \tag{3.6}$$

$$= \int_{0}^{\infty} f(x;\alpha,\beta)\mathrm{d}x + \int_{0}^{\infty} f(x;\alpha,-\beta)\mathrm{d}x.$$

Using (3.2) and Fubini theorem:

$$\int_0^\infty f(x;\alpha,\beta)\mathrm{d}x = \int_0^\infty \Big(\frac{1}{\pi}\int\limits_{-\beta/\alpha}^{\pi/2} \frac{\partial}{\partial x}\Big[-e^{-x^{\alpha/(\alpha-1)}U(\varphi;\beta)}\Big]\mathrm{d}\varphi\Big)\mathrm{d}x = \frac{1}{\pi}\int\limits_{-\beta/\alpha}^{\pi/2}\mathrm{d}\varphi\int_0^\infty \frac{\partial}{\partial x}\Big[-e^{-x^{\alpha/(\alpha-1)}U(\varphi;\beta)}\Big]\mathrm{d}x =$$

$$= \frac{1}{\pi}\int\limits_{-\beta/\alpha}^{\pi/2}\mathrm{d}\varphi = \frac{1}{2}+\frac{\beta}{\alpha\pi}$$

and

$$\int_0^\infty f(x;\alpha,-\beta)\mathrm{d}x = \int_0^\infty \Big(\frac{1}{\pi}\int\limits_{\beta/\alpha}^{\pi/2} \frac{\partial}{\partial x}\Big[-e^{-x^{\alpha/(\alpha-1)}U(\varphi;-\beta)}\Big]\mathrm{d}\varphi\Big)\mathrm{d}x = \frac{1}{\pi}\int\limits_{\beta/\alpha}^{\pi/2}\mathrm{d}\varphi = \frac{1}{2}-\frac{\beta}{\alpha\pi}.$$

The distribution function of stable distribution is:
for $x>0$

$$F(x;\alpha,\beta) = \int_{-\infty}^x f(y;\alpha,\beta)\mathrm{d}y = 1-\int_x^\infty f(y;\alpha,\beta)\mathrm{d}y$$

where

$$\int_x^\infty f(y;\alpha,\beta)\mathrm{d}y = \int_x^\infty \Big(\frac{1}{\pi}\int\limits_{-\beta/\alpha}^{\pi/2} \frac{\partial}{\partial y}\Big[-e^{-y^{\alpha/(\alpha-1)}U(\varphi;\beta)}\Big]\mathrm{d}\varphi\Big)\mathrm{d}y = \tag{3.7}$$

$$= \frac{1}{\pi}\int\limits_{-\beta/\alpha}^{\pi/2}\mathrm{d}\varphi\int_x^\infty \frac{\partial}{\partial y}\Big[-e^{-y^{\alpha/(\alpha-1)}U(\varphi;\beta)}\Big]\mathrm{d}y = \frac{1}{\pi}\int\limits_{-\beta/\alpha}^{\pi/2} e^{-x^{\alpha/(\alpha-1)}U(\varphi;\beta)}\mathrm{d}\varphi$$

and so

$$F(x;\alpha,\beta) = 1-\frac{1}{\pi}\int\limits_{-\beta/\alpha}^{\pi/2} e^{-x^{\alpha/(\alpha-1)}U(\varphi;\beta)}\mathrm{d}\varphi. \tag{3.8}$$

For $x<0$

$$F(x;\alpha,\beta) = \int_{-\infty}^x f(y;\alpha,\beta)\mathrm{d}y = \int_{-\infty}^x f(-y;\alpha,-\beta)\mathrm{d}y = \int_{|x|}^\infty f(y;\alpha,-\beta)\mathrm{d}y = \tag{3.9}$$

$$= \frac{1}{\pi}\int\limits_{\beta/\alpha}^{\pi/2} e^{-|x|^{\alpha/(\alpha-1)}U(\varphi;-\beta)}\mathrm{d}\varphi = 1-F(-x;\alpha,-\beta).$$

We simply deduce that $F(x;\alpha,\beta)+F(-x;\alpha,-\beta)=1$.

For $x=0$:

$$F(0;\alpha,\beta) = \frac{1}{2}-\frac{\beta}{\alpha\pi} \tag{3.10}$$

4 Representations via Expansions

In this section we derive aproximative formulas for stable densities and stable distribution functions.

Starting with relation (2.3), the integrand can be continously extended into complex plain and we denote it as $g(z; p_1)$. We see that complex valued function $g(z; p_1)$ is holomorphic. For $0 < r_1 < r_2 < \infty$ we define contour L as:

$$
\begin{aligned}
&L = L_1 \dotplus L_2 \dot{-} L_3 \dot{-} L_4, \text{ where} && (4.1)\\
&L_1(t) = t \text{ for } t \in [r_1, r_2];\ L_2(t) = r_2 e^{it} \text{ for } t \in [0, (p_1 - p_2)\frac{\pi}{2}];\\
&L_3(t) = te^{i(p_1-p_2)\frac{\pi}{2}} \text{ for } t \in [r_1, r_2];\ L_4(t) = r_1 e^{it} \text{ for } t \in [0, (p_1 - p_2)\frac{\pi}{2}].
\end{aligned}
$$

Let us integrate $g(z; p_1)$ along contour L:

$$
\begin{aligned}
\int_L e^{-izx - z^\alpha e^{-i\alpha(p_1-p_2)\frac{\pi}{2}}} \mathrm{d}z = &\int_{r_1}^{r_2} \exp\{-itx - t^\alpha e^{-i\alpha(p_1-p_2)\frac{\pi}{2}}\}\mathrm{d}t +\\
&+ i\int_0^{(p_1-p_2)\frac{\pi}{2}} r_2 \exp\{-ir_2 e^{it} x - r_2^\alpha e^{i\alpha t - i\alpha(p_1-p_2)\frac{\pi}{2}}\} e^{it}\mathrm{d}t -\\
&- \int_{r_1}^{r_2} e^{i(p_1-p_2)\frac{\pi}{2}} \exp\{-ite^{i(p_1-p_2)\frac{\pi}{2}} x - t^\alpha\}\mathrm{d}t -\\
&- i\int_0^{(p_1-p_2)\frac{\pi}{2}} r_1 \exp\{-ir_1 e^{it} x - r_1^\alpha e^{i\alpha t - i\alpha(p_1-p_2)\frac{\pi}{2}}\} e^{it}\mathrm{d}t.
\end{aligned}
$$

Because contour L is closed and integrand $g(z, p_1)$ is holomorphic, we can apply Cauchy Theorem which gives us:

$$
\int_L e^{-izx - z^\alpha e^{-i\alpha(p_1-p_2)\frac{\pi}{2}}} \mathrm{d}z = 0.
$$

For $r_1 \to 0,\ r_2 \to \infty$, the integrals along arcs $L_2,\ L_4$ converges to 0 and we get following equity:

$$
\int_0^\infty e^{-itx - t^\alpha e^{-i\alpha(p_1-p_2)\frac{\pi}{2}}} \mathrm{d}t = \int_0^\infty e^{i(p_1-p_2)\frac{\pi}{2}} e^{-ite^{i(p_1-p_2)\frac{\pi}{2}} x - t^\alpha} \mathrm{d}t. \tag{4.2}
$$

Using above equity, we can express stable density in (2.3) as:

$$
f(x; \alpha, p_1) = \frac{1}{\pi} \mathrm{Re} \int_0^\infty e^{i(p_1-p_2)\frac{\pi}{2}} e^{-ite^{i(p_1-p_2)\frac{\pi}{2}} x - ct^\alpha} \mathrm{d}t. \tag{4.3}
$$

Let us substitute $t = y^{1/\alpha}$:

$$
f(x; \alpha, p_1) = \frac{1}{\alpha\pi} \mathrm{Re} \int_0^\infty e^{i(p_1-p_2)\frac{\pi}{2}} e^{-iy^{1/\alpha} e^{i(p_1-p_2)\frac{\pi}{2}} x} e^{-cy} y^{\frac{1-\alpha}{\alpha}} \mathrm{d}y \tag{4.4}
$$

and expand $\exp\{-iy^{1/\alpha}e^{i(p_1-p_2)\frac{\pi}{2}}x\}$ into series:

$$e^{-iy^{1/\alpha}e^{i(p_1-p_2)\frac{\pi}{2}}x} = \sum_{k=0}^{\infty}\frac{[-iy^{1/\alpha}e^{i(p_1-p_2)\frac{\pi}{2}}x]^k}{k!} = \sum_{k=0}^{\infty}\frac{y^{k/\alpha}e^{-ip_2\pi k}x^k}{k!}.$$

We use linearity of integral and rewrite (4.4) as:

$$\begin{aligned} f(x;\alpha,p_1) &= \frac{1}{\alpha\pi}\mathrm{Re}\sum_{k=0}^{\infty}\frac{x^k e^{-ip_2\pi k+i(p_1-p_2)\frac{\pi}{2}}}{k!}\int_0^{\infty} y^{\frac{k+1}{\alpha}-1}e^{-cy}\mathrm{d}y = \\ &= \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{x^k\cos(-p_2\pi k+(p_1-p_2)\frac{\pi}{2})\Gamma(\frac{k+1}{\alpha}+1)}{(k+1)!c^{(k+1)/\alpha}} = \\ &= \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sin(p_2\pi k)\Gamma(k/\alpha+1)}{k!c^{k/\alpha}}x^{k-1}. \end{aligned}$$

Using the relation we can express later sum via gamma functions as:

$$f(x;\alpha,p_1) = \sum_{k=1}^{\infty}\frac{\Gamma(k/\alpha+1)}{\Gamma(k+1)\Gamma(p_2k)\Gamma(1-p_2k)}\frac{x^{k-1}}{c^{k/\alpha}}. \tag{4.5}$$

To determine expansion for stable distribution function, we can can integrate previous series:

$$\begin{aligned} F(x;\alpha,p_1)-F(0;\alpha,p_1) &= \int_0^x f(y;\alpha,p_1) = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sin(p_2\pi k)\Gamma(k/\alpha+1)}{k!kc^{k/\alpha}}x^k = \\ &= \sum_{k=1}^{\infty}\frac{\Gamma(k/\alpha+1)}{\Gamma(k+1)\Gamma(p_2k)\Gamma(1-p_2k)}\frac{x^k}{kc^{k/\alpha}} \end{aligned}$$

and so:

$$\begin{aligned} F(x;\alpha,p_1) &= p_2 + \sum_{k=1}^{\infty}\frac{p_2\Gamma(k/\alpha+1)}{\Gamma(k+1)\mathrm{B}(1+p_2k,1-p_2k)}\frac{x^k}{c^{k/\alpha}} = \\ &= p_2\sum_{k=0}^{\infty}\frac{\Gamma(k/\alpha+1)}{\Gamma(k+1)\mathrm{B}(1+p_2k,1-p_2k)}\frac{x^k}{c^{k/\alpha}}. \end{aligned}$$

Let us summerize the results into the lemma:

Lemma 4.1. *For stable density $f(x;\alpha,p_1)$ and $x>0$ the following holds:*

$$f(x;\alpha,p_1) = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sin(p_2\pi k)\Gamma(k/\alpha+1)}{k!c^{k/\alpha}}x^{k-1}. \tag{4.6}$$

Let us note that first part of the lemma is summerized Feller XVII.7 Lemma 1, p.583 in [1]. The method how the result is derived is different from the one presented.

5 Incomplete Gammma Function

Recall that Gamma function, a natural extension of factorial, is defined by integral:

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} \mathrm{d}x, \tag{5.1}$$

where $a \in \mathbb{C}, \mathcal{R}a > 0$.

Having space of test functions $\mathcal{S}$, $\varphi(x) = e^{-x} \in \mathcal{S}$ for $x \in \mathbb{R}^+$ and so we can represent gamma function as:

$$\Gamma(a) = \langle x^{\alpha-1} 1_{\{x>0\}}, e^{-x} \rangle. \tag{5.2}$$

A very natural idea is investigation of integrals:

$$\Gamma(a) = \int_0^x y^{a-1} e^{-y} \mathrm{d}y + \int_x^\infty y^{a-1} e^{-y} \mathrm{d}y, \tag{5.3}$$

where we denote

$$\gamma(a,x) = \int_0^x y^{a-1} e^{-y} \mathrm{d}y, \tag{5.4}$$

$$\Gamma(a,x) = \int_x^\infty y^{a-1} e^{-y} \mathrm{d}y, \tag{5.5}$$

and call functions $\Gamma(a,x), \gamma(a,x)$ incomplete gamma functions.

Expansion for incomplete gamma function:

$$\Gamma(a,x) = (a-1)! e^{-x} \sum_{k=0}^{a-1} \frac{x^k}{k!}, k > 0. \tag{5.6}$$

Lemma 5.1. *Random variable X with Poisson distribution with parameter $a > 0$ has probability distribution in terms of incomplete Gamma function as follows:*

$$P\{X \le x\} = \frac{\Gamma(a+1,x)}{\Gamma(a+1)}. \tag{5.7}$$

Lemma 5.2. *Random variable X with standart Gaussian distribution has a probability distribution in terms of incomplete Gamma function as follows:*

$$P\{X \le x\} = \begin{cases} \frac{\Gamma(\frac{1}{2}, \frac{x^2}{2})}{2\sqrt{\pi}} & \text{for } x \le 0, \\ 1 - \frac{\Gamma(\frac{1}{2}, \frac{x^2}{2})}{2\sqrt{\pi}} & \text{for } x > 0. \end{cases} \tag{5.8}$$

Proof. Starting from Gaussian probability distribution function we use substitution to get expression via incomplete gamma function. Assume first $x > 0$:

$$P\{X \le x\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} \mathrm{d}y = 1 - \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{y^2}{2}} \mathrm{d}y$$

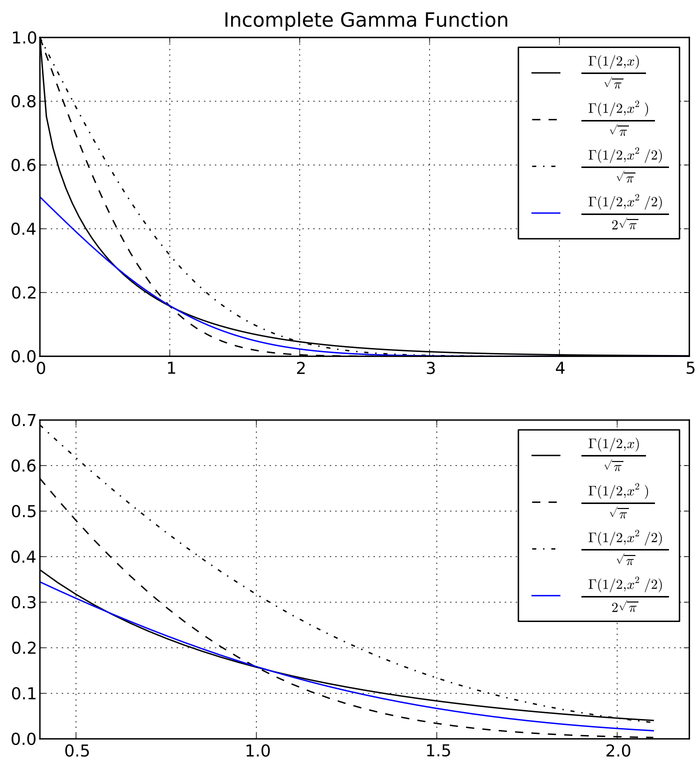

Figure 1: Incomplete Gamma Function

Substitute $z = \frac{y^2}{2}$, then $\mathrm{d}y = \frac{\mathrm{d}z}{\sqrt{2z}}$ and:

$$P\{X \leq x\} = 1 - \frac{1}{2\sqrt{\pi}} \int_{x^2/2}^{\infty} z^{-\frac{1}{2}} e^{-z} \mathrm{d}z = 1 - \frac{\Gamma(\frac{1}{2}, \frac{x^2}{2})}{2\sqrt{\pi}}.$$

For $x \leq 0$

$$P\{X \leq x\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} \mathrm{d}y = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{y^2}{2}} \mathrm{d}y = \frac{\Gamma(\frac{1}{2}, \frac{x^2}{2})}{2\sqrt{\pi}}.$$

Q.E.D

Lemma 5.3. *Random variable X with Lévy distribution has a probability dis-*

tribution function in terms of incomplete Gamma function as follows:

$$P\{X \le x\} = \frac{\Gamma(\frac{1}{2}, \frac{1}{2x})}{\sqrt{\pi}}. \tag{5.9}$$

Proof.

$$P\{X \le x\} = \frac{1}{\sqrt{2\pi}} \int_0^x y^{-\frac{3}{2}} e^{-\frac{1}{2y}} \mathrm{d}y$$

Substitute $z = \frac{1}{2y}$, then $\mathrm{d}y = -\frac{\mathrm{d}z}{2z^2}$ and:

$$P\{X \le x\} = -\frac{1}{\sqrt{2\pi}} \int_\infty^{1/2x} \frac{(2z)^{\frac{3}{2}}}{2z^2} e^{-z} \mathrm{d}z = \frac{1}{\sqrt{\pi}} \int_{1/2x}^\infty z^{-\frac{1}{2}} e^{-z} \mathrm{d}z = \frac{\Gamma(\frac{1}{2}, \frac{1}{2x})}{\sqrt{\pi}}.$$

Q.E.D

Definition 5.1. Class of Compound Incomplete Gamma Functions:

$$P\{X \le x\} = g(x) \frac{\Gamma(\alpha, f(x))}{\Gamma(\alpha)} \tag{5.10}$$

Let us derive characteristic function:

$$\frac{\mathrm{d}}{\mathrm{d}x} \Gamma(\alpha, f(x)) \tag{5.11}$$

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